CHAPTER 4

Nonlinear Risks and the Treatment of Bonds and Options

In the previous chapter, we discussed risk measurement for single "well-behaved" securities, with returns that can appropriately be mapped to returns of one risk factor. In this basic case, the asset also has returns that move *linearly*, that is, one-for-one or in a constant proportion to some underlying asset or risk factor return. But many securities are not at all well-behaved in this sense. Rather, they have *nonlinear* returns that have much larger or smaller responses to some other asset returns, depending on the asset price level.

Nonlinearity can vitiate risk measurement techniques such as VaR that are designed primarily for linear exposures. In this chapter, we discuss nonlinearity, and how to measure risk in its presence. We'll focus on two important examples of nonlinear securities, options and bonds. Another reality that we will have to address is that many assets are complex, and are exposed to multiple risk factors, and that most real-world portfolios contain many positions and are exposed to multiple risk factors. In the next chapter, studying risk measurement for portfolios, we focus on assets that are sensitive to multiple risk factors.

Options and option-like exposures depart in both ways from the previous chapter's model: nonlinearity and dependence on several risk factors. First, the P&L of an option is a nonlinear function of returns on the underlying asset. A relatively small return on the underlying asset can have a large impact on option P&L. The P&L would therefore not be normally distributed, even if risk factor changes were. Options are not unique in this regard. Bond prices are also nonlinearly related to interest rates or yields. Bond traders refer to this nonlinear sensitivity as *convexity*.

Second, option returns depend jointly on several market risk factors, the underlying asset price, the financing or risk-free interest rate, and the underlying asset's dividend, interest yield, or storage cost. There is also a type of volatility risk that is peculiar to options and option-like securities.

Options are a species of derivatives. As noted in Chapter 1, derivatives are securities with values that are functions of the values of other assets or risk factors. But not all derivatives are nonlinear. There are two basic types of derivatives: futures, forwards, and swaps on the one hand, and options on the other.

Futures, forwards, and swaps have a linear and symmetric relationship to the underlying asset price and can be hedged statically. *Static hedging* means that the derivatives position can be hedged with a one-time trade in the underlying asset. This does not mean their values move one-for-one with those of the underlying, but rather that their responsiveness to changes in the underlying is constant.

The possibility of static hedging means that only the value of the underlying asset and not its volatility determines the value of the derivative, so futures, forwards, and swaps generally have zero *net present value* (NPV) at initiation. We note this here, since it is a direct consequence of linearity; it becomes important when we study counterparty credit exposure in Chapter 6.

Options have a nonlinear relationship to the underlying asset price and must be hedged dynamically to minimize their risks. *Dynamic hedging* means that the amount of the underlying asset that neutralizes changes in the derivative's value itself changes over time, so repeated trades are needed to stay hedged. For some values of the underlying asset, the option value may move close to one-for-one with the underlying, while for other values of the underlying it may hardly change at all. In general, volatility is an important element in the value of an option, so an option contract cannot have zero NPV at initiation.

Nonlinearity is also important because it is one of two ways that the P&L distribution can have fat tails, that is, a tendency toward very large positive and/or negative values:

- The payoff function of the security may lead to disproportionately large-magnitude returns for modest changes in the underlying asset price or risk factor. For example, a given change in the value of the underlying price may lead to a much larger or smaller change in the value of an option at different levels of the underlying. Nonlinearity is the focus of this chapter.
- The distribution of risk factor or asset price returns may be non-normal. We discuss this possibility in more detail in Chapter 10.

4.1 NONLINEAR RISK MEASUREMENT AND OPTIONS

A number of approaches to modeling option risk have been developed. We'll focus on two relatively simple approaches. The first applies the simulation techniques we developed in Chapter 3 in a way that takes account of the special problems raised by the nonlinearity of options. The second, called *delta-gamma*, uses a quadratic approximation to option returns. We'll discuss these for plain-vanilla options, but they can be applied to exotic options, convertible bonds, and other structured products. Both techniques can help address the difficulties in accurately measuring risk generated by nonlinearity, but neither completely solves them. After describing the delta-gamma technique for options, we'll show how it can be applied to fixed-income securities.

This section assumes some, but not terribly much, familiarity with option pricing models, and readers can see the textbooks cited at the end of the chapter to brush up. In this chapter, we simplify things by talking about options in the context of the standard Black-Scholes-Merton pricing model. We denote the Black-Scholes theoretical or model value of a plain-vanilla *European call option* by $v(S_t, T-t, X, \sigma, r, q)$ and that of a *put option* by $w(S_t, T-t, X, \sigma, r, q)$,

where $S_t =$ is the time-t underlying price
$\sigma =$ is the time-t asset return volatility
$X =$ is the exercise price
$T =$ is the maturity date, and $\tau = T - t$ the time to maturity
$r =$ is the financing rate.
$q =$ is the cash flow yield, such as a dividend or coupon, on the underlying asset.

A European option is one that can only be exercised at maturity, in contrast to an *American option*, which can be exercised anytime prior to expiration. We define "one option" as an option on one unit—share, ounce, or currency unit—of the underlying asset.

The formulas are spelled out in Appendix A.3. Option value depends on its "design parameters," the things that are part of the option contract: whether it is a put or a call, its time to maturity, and the exercise price. It also depends on market risk factors: the underlying asset price, the financing or risk-free interest rate, the underlying asset's cash-flow yield, and the asset return volatility. The cash flow yield can be the interest paid by a foreign-currency bank deposit, the coupon yield of a bond, a negative rate representing the cost of storing oil or gold, or the dividend yield paid by common stock.

In the Black-Scholes model, we assume that interest rates and the cash flow yield are constant and nonrandom. Most importantly, we assume that the return volatility σ of the underlying asset is constant, nonrandom, and that we have a reliable estimate of it, so the risk of the option is related only to fluctuations in the underlying asset price. Let's highlight two important aspects of the constant-volatility assumption. First, in the last chapter, we treated return volatility as a time-varying quantity, and sought to obtain an accurate short-term forecast of its future value, conditional on recent return behavior. Here, we treat volatility as constant. If this were only true, we would, after a relatively short time interval, be able to estimate the volatility with near-perfect accuracy. It would then matter little whether we treat σ as a known or an estimated parameter.

Second, note that the volatility parameter is the only one that can't be observed directly. In many applications, we take σ to be the *implied volatility*, that is, an estimate of volatility that matches an observed option price to the Black-Scholes formula, given the observed values of the remaining arguments, S_t, τ, X, r, and q. But there are crucial gaps between the Black-Scholes model and actual option price behavior. Implied volatility fluctuates over time, and not only because the conditional volatility of the underlying is time-varying. Market participants' desire to buy and sell options fluctuates, too, because of their changing appetite for risk in general, because their hedging needs change over time, and because of their changing views on future returns on the underlying asset, among other reasons. Implied volatility risk, which we discuss in Chapters 5 and 10, is a key risk factor for option positions.

Staying in the Black-Scholes world for now will help us discuss the risk measurement issues arising from nonlinearity. The volatility σ is then both the actual and implied volatility. For the rest of this chapter, the risk of the option is understood to be driven by the risk of the underlying asset, that is, changes in S_t, alone.

For concreteness, let's look at a foreign-exchange option, specifically, a European call on the euro, denominated in U.S. dollars, struck *at-the-money forward*, with an initial maturity of one week. The long option position is unhedged, or "naked." The spot rate at initiation is \$1.25 per euro, the domestic (U.S.) one week money market rate (r) is 1 percent, and the euro deposit rate (q) is 28 basis points, both per annum. "At-the-money forward" means that the strike price of the option is set equal to the current one-week forward foreign exchange rate. This is a standard way of setting option strikes in the OTC currency option markets. Given the domestic and foreign deposit rates, the one-week forward rate, the exercise price of the option, is slightly higher than the spot rate; the euro trades at a small premium of about two ticks, that is, the forward rate "predicts" a slight dollar depreciation to \$1.2502 over the subsequent week. This is consistent with *covered interest rate parity* and the absence of forward arbitrage, since the domestic interest rate is slightly higher than the foreign one. We assume finally, that the actual and implied volatility σ is 12 percent per annum.

In the Black-Scholes model, logarithmic exchange rate returns follow the probability distribution:[1]

$$\log(S_{t+\tau}) \sim N\left[\log(S_t) + \left(r - q - \frac{\sigma^2}{2}\right)\tau, \sigma\sqrt{\tau}\right]$$

This distributional hypothesis is consistent with the parametric VaR examples of Chapter 3. The mean of the time-$T = t + \tau$ spot rate is equal to the forward rate, and its standard deviation is equal to the constant volatility, adjusted for the time horizon $t + \tau$. Just as in Chapter 3, we have set the parameters in our example so that the drift over discrete time periods is equal to zero:

$$r - q = \frac{\sigma^2}{2} = 0.0072$$

In the case of forward foreign exchange, under covered interest parity, the drift is equal to the spread between the two interest or deposit rates involved, minus the volatility adjustment. In our example, we've set the spread so as to zero out the drift. The value of a one-week European call is then \$0.00838, or a bit more than $\frac{8}{10}$ of a U.S. cent for each euro of underlying notional amount.

4.1.1 Nonlinearity and VaR

Now that we've set up the model and the example, let's start by imagining the simplest possible way to compute a one-day VaR for the one-week European call on the euro. The upper panel of Figure 4.1 shows the P&L of a long position in one euro call after one day as a function of the underlying exchange rate, including one day of *time decay*, the perfectly predictable change in value of the option as its remaining maturity is shortened by the passing of time. The 1 and 99 percent quantiles of the one-day ahead exchange rate, given the posited probability distribution, are marked in the graph with vertical grid lines. The points at which those grid lines intersect the P&L plot mark the 1 and 99 percent quantiles of the next-day P&L.

[1]See Appendix A.3 for more detail on this distribution and its relationship to the Black-Scholes model.

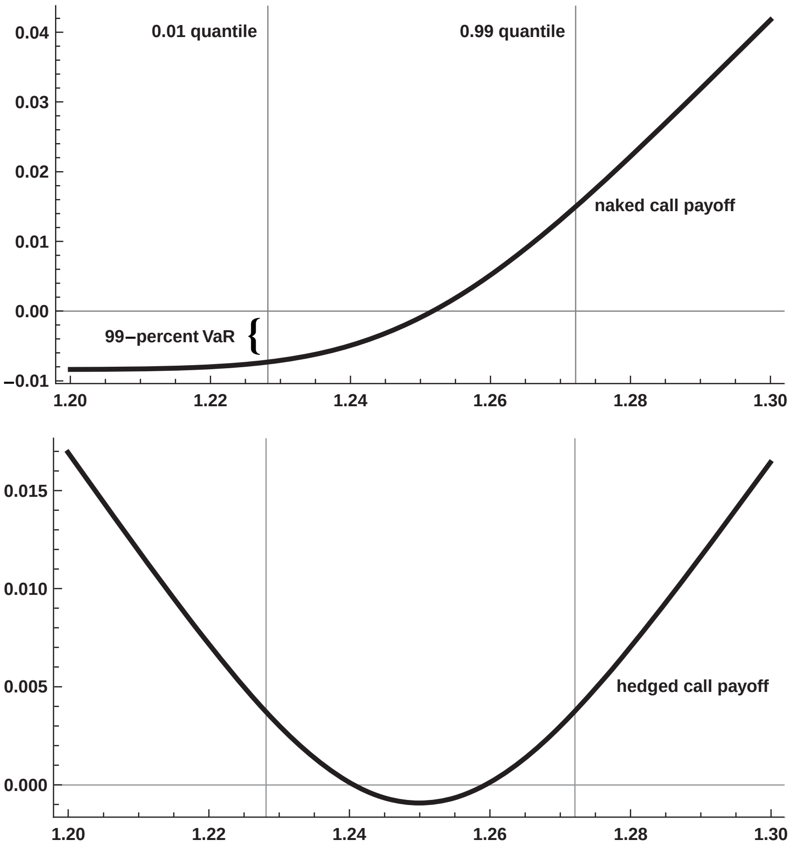

FIGURE 4.1 Monotonicity and Option Risk Measurement
One-day P&L in dollars of an at-the-money forward call on the euro, denominated in U.S. dollars, as a function of the next-day USD-EUR exchange rate. The option parameters and the spot rate's probability distribution are as described in the text. Vertical grid lines denote the 1st and 99-th percentiles of the exchange rate.
Upper panel: One-day P&L of a position in one euro call as a function of the underlying exchange rate.
Lower panel: One-day P&L of a position in one euro call, delta-hedged at the beginning of the period, as a function of the underlying exchange rate.

If the value of a position is a nonlinear function of a single risk factor, and we know or stipulate its P&L distribution, it looks fairly straightforward to compute any of its P&L quantiles, including the VaR, directly from the P&L distribution. The 99 percent VaR of the call position in our example is equal to about \$0.0065.

Let's try to generalize this approach, which we'll refer to as "analytical," to other types of option positions. We denote the pricing or value function of the position or security by $f(S_t, \tau)$, where S_t is the sole risk factor. For the specific case of a European option, we set $f(S_t, \tau) = v(S_t, \tau, X, \sigma, r, q)$ or $f(S_t, \tau) = w(S_t, \tau, X, \sigma, r, q)$, since σ is a parameter for now, and S_t the only random driver of value. The number of options in the position is denoted x, so the time-t value of the portfolio is $V_t = xf(S_t, \tau)$, and the random P&L over the VaR time horizon θ is

$$V_{t+\theta} - V_t = x\,[\,f(S_{t+\theta}, \tau - \theta) - f(S_t, \tau)]$$

The function arguments in the first term inside the square brackets reflect the fact that the VaR will be determined by the exchange rate one day ahead, and its impact on the value of an option with a maturity that is now one day shorter. We would like to be able to calculate, say, the one day, 99 percent VaR, by setting it equal to

$$f(S^*, \tau - \theta) - f(S_t, \tau)$$

per option, where S^* is the, say, 1st or 99-th percentile of S_t, depending on whether the position is long or short. The VaR shock, the asset return at which the VaR is just reached, is then

$$r^* = \log\left(\frac{S^*}{S_t}\right) \quad \Leftrightarrow \quad S^* = S_t e^{r^*}$$

But this only works if the position value or P&L is a *monotonic* or *monotone function* of the underlying risk factor. The function $f(S_t, \tau)$ is called *monotone increasing* in S_t if and only if

$$S_1 > S_2 \quad \Leftrightarrow \quad f(S_1, \tau) > f(S_2, \tau) \qquad \forall S_1, S_2$$

A function $f(S_t, \tau)$ is called *monotone decreasing* in S_t if and only if

$$S_1 > S_2 \quad \Leftrightarrow \quad f(S_1, \tau) < f(S_2, \tau) \qquad \forall S_1, S_2$$

Plain-vanilla options fulfill this requirement. The long call shown in the upper panel of Figure 4.1, for example, is monotone increasing; its slope or "delta" (to be defined in a moment) is never negative. A put is a nonincreasing function; its delta is never positive.

The lower panel of Figure 4.1 illustrates a case in which this "analytical" method of VaR calculation won't work, because the portfolio value is not a monotone function of the underlying asset price. It displays the next-day P&L on a *delta-hedged* position in the call. Here, the first percentile of the exchange rate doesn't tell us the VaR, because there is another, higher, exchange rate realization, close to the 99th percentile, that has *exactly the same P&L*.

The lower panel suggests that the VaR shock is zero or just a small fluctuation away from the initial exchange rate. The delta-hedged call is a "long volatility" or "long gamma" trade, and is most profitable if there are large exchange-rate fluctuations in either direction.

So there is a crucial additional requirement, monotonicity, if we are to take the simple "analytical" approach to risk measurement for an option position. Monotonicity also enters into the standard theory behind the distribution of a transformation of a random variable with a known distribution. This standard theory, discussed in Appendix A.5, requires that the *inverse function* of the transformation exist, and only monotone functions are "one-to-one" and thus invertible. Monotonicity is thus a requirement for being able to "pass through" the probability distribution of S_t to the probability distribution of $V_t - V_{t+\tau}$.

4.1.2 Simulation for Nonlinear Exposures

Since we can't assume the P&L is a monotonic function of the underlying asset, the "analytical" approach doesn't work in general. In the example above, monotonicity didn't hold even for a position exposed to a single risk factor. Very few portfolios are that simple, and the invertibility condition is very hard to meet in practice. An approach that uses the same model for the underlying return behavior, but doesn't fail in the face of nonmonotonicity, is *Monte Carlo* or *historical simulation with full repricing*. This procedure is similar to that laid out in Chapter 3. But we have here the additional step of computing the P&L via $f(S_t, \tau)$ in each simulation thread, rather than simply ascertaining the value of S_t in each simulation thread.

The first two steps of either of these simulation approaches are to prepare the simulated returns. These are identical to the first two steps of the Monte Carlo and historical simulation techniques as laid out for linear positions in Chapter 3. The difference for nonlinear positions is in how the simulated returns are treated next, in the "repricing" step. Instead of multiplying the position value by the simulated returns, we enter the simulated return into the pricing or value function $f(S_t, \tau)$.

For the Monte Carlo simulation technique, and still assuming the normal distribution, the simulated exchange rates are

$$\tilde{S}^{(i)} = S_t \exp\left(\sigma\sqrt{\tau}\tilde{\epsilon}^{(i)}\right) \qquad i = 1, \ldots, I$$

where $\tilde{\epsilon}^{(i)}, i = 1, \ldots, I$ are a set of independent draws from a $N(0, 1)$ distribution. We set the VaR time horizon $\theta = \frac{1}{252}$ and $I = 10{,}000$ in our examples.

The 99 percent VaR is found by substituting the $\tilde{S}^{(1)}, \ldots, \tilde{S}^{(I)}$ into the P&L function $x[f(S_{t+\theta}, \tau - \theta) - f(S_t, \tau)]$ to get

$$x\left[f(\tilde{S}^{(i)}, \tau - \theta) - f(S_t, \tau)\right] \qquad i = 1, \ldots, I$$

and taking the 1st percentile, or 10th worst outcome. The result in our example is a VaR estimate of \$0.0065, or about $\frac{65}{100}$ of one cent per euro of underlying notional, and about the same as that obtained with the "analytical" approach. The result is not identical to that of the "analytical" approach because of simulation noise: the distribution of the $\tilde{S}^{(i)}$ is only approximately equal to that of S_t.

Monte Carlo with full repricing is often impractical for computing risk for large portfolios. It can be slow if there are many positions, and if enough of those positions are priced via complicated models, though this is becoming less of a problem as computing power increases. Many derivative and structured credit pricing models are implemented via simulation rather than analytical solution. That means that if we want to know what the security is worth for a given value of the underlying asset or other market inputs, we don't plug the $\tilde{S}^{(i)}$ into a formula, but rather simulate the security value using the inputs as parameters. We give a detailed example of such a procedure in Chapter 9. In many cases, each simulation of a security value can be quite "expensive" in computer time. In order to do risk computations with $I = 10{,}000$ for such an asset, we must repeat this pricing process 10,000 times. If each repricing requires, say, 1,000 simulations to be accurate, a total of 10,000,000 simulation threads have to be computed. Even a great deal of sheer computing power may not speed it up enough to be practicable.

But if we stipulate the distribution of the underlying returns, Monte Carlo can be made as accurate as we like, as long we use as much computing time and as many simulations as are needed for that level of accuracy and the pricing model we are using. It therefore serves as the typical benchmark for assessing the accuracy of other approaches, such as delta-gamma, which we discuss next. Of course, to use Monte Carlo as a benchmark, you have to run it, at least in studies, if not in practice.

4.1.3 Delta-Gamma for Options

So far, we have developed a simple approach that fails in the absence of monotonicity, and a simulation approach that is too slow to be practicable in general. The next approach we will explore, delta-gamma, does not help at all with the monotonicity problem, but it can speed up calculations compared to full repricing. It raises new statistical issues, and, more importantly, it is based on an approximation that hedging can be wildly wrong. But when delta-gamma is safe to use, it can be very useful.

The starting point for the delta-gamma approach is to approximate the exposure of a call to fluctuations in the underlying asset price, using a quadratic approximation in S_t, by

$$\begin{aligned}\Delta v(S_t, \tau, X, \sigma, r, q) &= v(S_t + \Delta S, \tau - \Delta t, X, \sigma, r, q) \\ &\quad - v(S_t, \tau, X, \sigma, r, q) \\ &\approx \theta_{c,t}\Delta t + \delta_{c,t}\Delta S + \frac{1}{2}\gamma_t \Delta S^2\end{aligned} \tag{4.1}$$

The *theta* of the call option is

$$\theta_{c,t} \equiv \frac{\partial}{\partial t} v(S_t, \tau, X, \sigma, r, q)$$

The theta of a long position in a put or call option is negative, and represents the predictable loss of option value as its maturity shortens.

The *delta* of the option, the slope of the option pricing function we referred to above, is

$$\delta_{c,t} \equiv \frac{\partial}{\partial S_t} v(S_t, \tau, X, \sigma, r, q)$$

The delta is also the amount of the underlying asset that must be bought or sold to hedge the option against small fluctuations in the underlying price. Corresponding definitions can be provided for a put. The *gamma* is

$$\gamma_t \equiv \frac{\partial^2}{\partial {S_t}^2} v(S_t, \tau, X, \sigma, r, q)$$

and is the same for both a call and put with the same strike and maturity. The delta and gamma change with the underlying price, the implied volatility, and the other market and design parameters of the options, but we are not spelling out this dependency in the notation, except to put a time subscript on the sensitivities.

We have two ways now to use the quadratic approximation to compute the one day, 99 percent VaR:

- "Analytical," that is, find the quantile of the future exchange rate corresponding to the 0.01 quantile of the P&L function, and substitute it into the delta-gamma approximation to the P&L function. This approach will not solve the monotonicity problem, but it is fast.

- Simulation, that is, use the same simulated values of the future exchange rate as in full repricing, but use the delta-gamma approximation to compute each simulation thread's P&L. The first percentile (times −1) is the VaR. This approach *will* solve the monotonicity problem, and it is much faster than full repricing. But it can be inaccurate, as we will see in a moment.

We can also compare the quadratic to a linear approximation using delta alone:

$$\Delta v(S_t, \tau, X, \sigma, r, q) \approx \theta_{c,t}\Delta t + \delta_{c,t}\Delta S$$

As we will see, there are tricky issues in choosing between a linear and higher-order approximation.

Let's look at examples that cover a range of option position types frequently encountered in practice. For each one, we'll graph the P&L function itself as a solid curve, and the delta (linear) and delta-gamma (quadratic) approximations to the P&L function. For each option prtfolio, we will also display the histogram of simulated P&Ls using the P&L function itself and the delta-gamma approximation. The one-day, 99 percent confidence level VaR results are summarized in Table 4.1 for the three option position types and the three approaches to VaR estimation.

Unhedged Long Call The upper panel of Figure 4.2 illustrates the delta-gamma approximation for an unhedged, or "naked," long call. The solid plot shows the one-day change in value of the one week euro call of our standing example as the spot exchange rate varies.

The analytical approach can be used, since the payoff profile is monotonic. The analytical as well as the simulation approaches using the delta-gamma approximation or full repricing all give roughly equal VaR estimates of about \$0.0073. Any of these are much more accurate than the delta approximation, which substantially overestimates the VaR.

TABLE 4.1 Comparison of Option VaR Estimates

	Analytical	Delta	Delta-Gamma	Full Repricing
Long call	0.00731	0.01178	0.00736	0.00726
Delta-hedged long call	NA	0.00092	0.00092	0.00092
Risk reversal	0.01178	0.01073	0.00410	0.01152

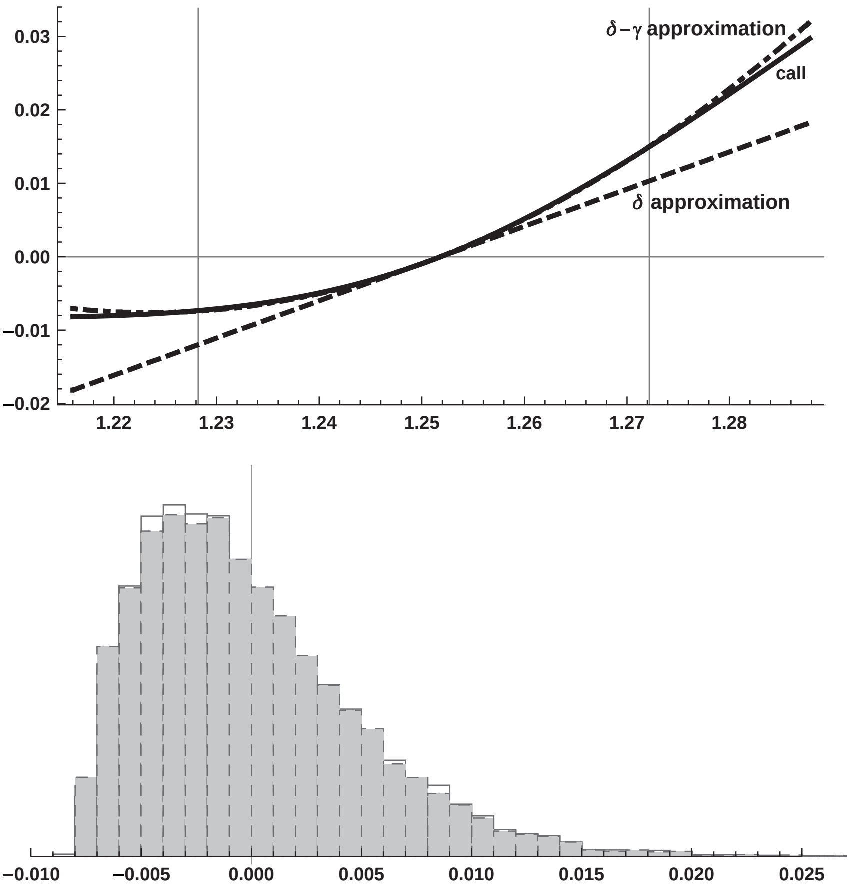

FIGURE 4.2 Delta-Gamma and Full-Repricing VaR for an Unhedged Long Call
Upper panel: One-day P&L in dollars of a long call option position as a function of the next-day USD-EUR exchange rate. The solid curve plots the exact P&L function, the dashed curve the delta approximation, and the dot-dashed curve the delta-gamma approximation.
Lower panel: Histogram of simulated P&Ls using the exact P&L function (unshaded bars), and the delta-gamma approximation (shaded bars).

Hedged Long Call For a hedged long call, we can't use the analytical approach at all, since the P&L function is notmonotone. The hedged call has a delta of zero when the hedge is first put on, so the quadratic approximation becomes

$$\Delta v(S_t, T-t, X, \sigma, r, q) \approx \theta_{c,t}\Delta t + \frac{1}{2}\gamma_t \Delta S^2$$

As can be seen from Figure 4.3, the worst losses on this long gamma strategy—a strategy for which the position gamma is positive—occur when the exchange rate doesn't move at all. A short gamma position has its worst losses when there is a large exchange-rate move. This is an important point: A delta-hedged option position still has a potentially large gamma exposure to the underlying price, since it cannot be hedged continuously in time. Rather, the hedge is rebalanced at discrete intervals.

At the 1st and 99th percentiles of the exchange rate's next-day distribution, the P&L of the long gamma trade is positive. The nonrandom one day time decay of \$0.00092 is the largest possible loss, and is offset to a greater or lesser extent by gamma gains from any exchange rate fluctuations. About two-thirds of the simulation scenarios are losses, but within the narrow range $(-0.00092, 0)$.

All three simulation approaches give almost exactly the same result. The reason is that a small loss, very close to the time decay of the option, is the likeliest outcome. Under the normal distribution, most fluctuations are small, and the gamma is therefore a very small positive number. So the lower quantiles of the P&L distribution are equal to the time decay, minus a tiny gamma gain.

Option Combinations Before describing the next option position, involving several options, we need to introduce some terminology. An *option combination* is a portfolio of options containing both calls and puts. An *option spread* is a portfolio containing only calls or only puts. A combination or a spread may contain either short or long option positions, or both.

One of the most common option combinations is the *straddle*, consisting of a call and a put, both either long or short, both struck at-the-money spot or forward, and both with the same maturity. In the options on futures markets, the exercise price is generally chosen to be close to the price of the current futures with the same expiration date as the options.

Almost as common are combinations of out-of-the-money options, particularly the *strangle* and the *risk reversal*. Both consist of an out-of-the-money call and out-of-the-money put. In these two combinations, the exercise price of the call component is higher than the current spot or forward asset price (or futures), and the exercise price of the put is lower. In a risk reversal, an out-of-the-money call is exchanged for an out-of-the-money put, with a net premium paid by one to the other counterparty. In a strangle, one counterparty sells both an out-of-the-money call and an out-of-the-money put to the other. Figure 4.4 displays the payoff profiles of these combinations.

In the OTC foreign exchange option markets, risk reversals and strangles are usually standardized as combinations of a 25-delta call and a 25-delta put (or 10-delta). The exercise prices of the call and put are both set so

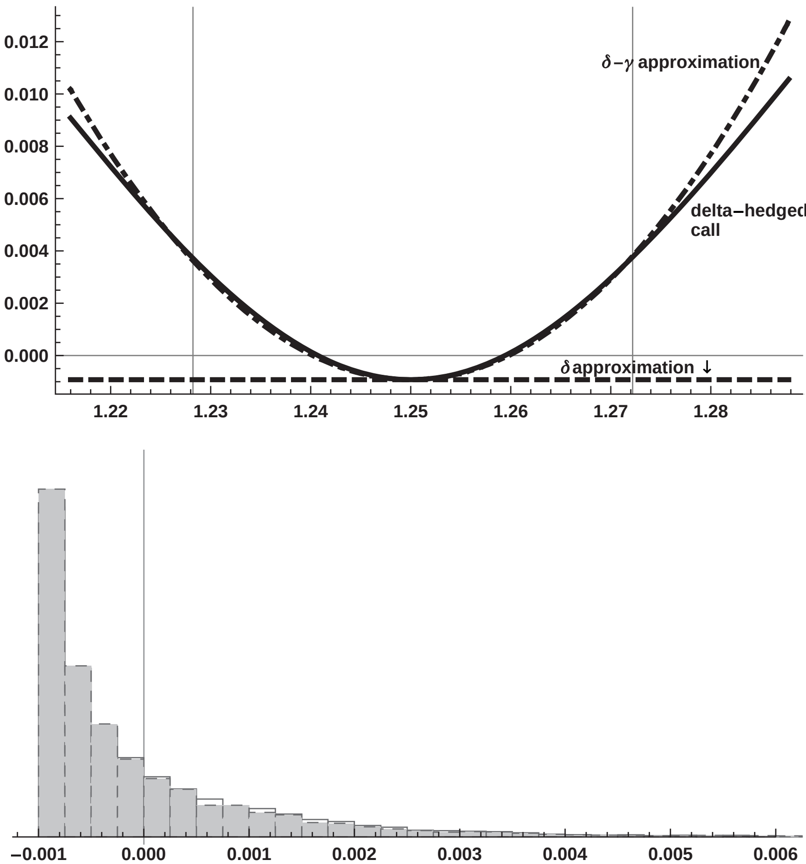

FIGURE 4.3 Delta-Gamma and Full-Repricing VaR for a Hedged Call
Upper panel: One-day P&L in dollars of a delta-hedged long call option position as a function of the next-day USD-EUR exchange rate. The solid curve plots the exact P&L function, the dashed curve the delta approximation, and the dot-dashed curve the delta-gamma approximation.
Lower panel: Histogram of simulated P&Ls using the exact P&L function (unshaded bars), and the delta-gamma approximation (shaded bars).

that their forward deltas are equal to 0.25 or 0.10. We describe the quoting conventions of these instruments in more detail in Section 5.5.

Risk reversals provide a good illustration of the relationship between nonlinearity in the pricing function and the pitfalls of relying on linear or quadratic approximations to the pricing function to measure risk. The risk reversal's P&L function is monotone, but alternates between concave and convex segments. This renders the delta-gamma approach much less accurate; in fact, it is significantly worse than using delta alone, as can be seen

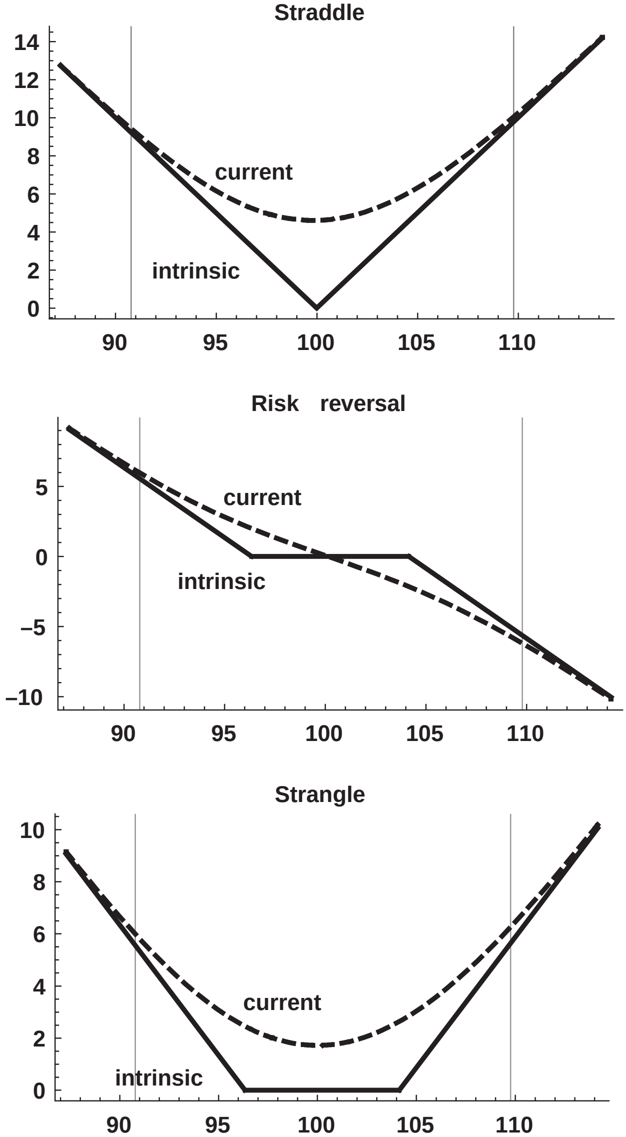

FIGURE 4.4 Option Combinations
Each panel shows the intrinsic and current values of an option combination. The current price of the underlying asset is 100, the annual implied volatility is 20 percent, the time to maturity of all the options is 1 month, and the risk-free rate and dividend rate on the underlying asset are both set at 1 percent. The forward asset price is therefore also 100. In each panel, the solid line represents the intrinsic and the dashed line the current value of the combination as a function of the terminal or current price of the underlying. The vertical grid lines mark the 1 and 99 percent quantiles of the risk-neutral distribution of the underlying asset.

in Figure 4.5. Surprisingly, using the "less accurate" linear approximation is in fact *less* misleading than using the quadratic approximation, which underestimates the VaR by over one-half.

The risk reversal is similar to a delta-hedged option in that it has an initial net delta that is close to zero. Its initial gamma is much lower, though, since there are two offsetting options; the long option position has positive and the short option has negative gamma. Moreover, each of the component options has a lower-magnitude gamma at the initial exchange rate than an at-the-money option with the same maturity.

An alternative approach to VaR measurement would be to compute the variance of the change in value of the option $\Delta v(S_t, T-t, X, \sigma, r, q)$. In other words, we would treat the option value itself, rather than the underlying asset price, as the risk factor. But this involves an additional statistical issue. The quadratic approximation gives us $\Delta v(S_t, T-t, X, \sigma, r, q)$ as a function of the change in the value of the underlying asset and the squared change. To compute its variance, we would have to take into account the joint distribution of a normal variate, and its square. Other analytical approximations to delta-gamma have also been proposed, but none appear to be decisively superior in accuracy over a wide range of position types and asset classes.

4.1.4 The Delta-Gamma Approach for General Exposures

The delta-gamma approach can be applied to any security with payoffs that are nonlinear in a risk factor. For simplicity, let's continue to look at securities that are functions $f(S_t, \tau)$ of a single risk factor S_t and time. Just as we did for an option in Equation (4.1), we can approximate changes in value by the second-order Taylor approximation:

$$\Delta f(S_t, \tau) = f(S_t + \Delta S, \tau - \Delta t) - f(S_t, \tau) \approx \theta_t \Delta t + \delta_t \Delta S + \frac{1}{2}\gamma_t \Delta S^2$$

The *theta* of the security is the perfectly predictable return per time period:

$$\theta_t \equiv \frac{\partial f(S_t, \tau)}{\partial \tau}$$

For a bond or a dividend-paying stock, for example, the theta is the coupon or other cash flow, and is a positive number.

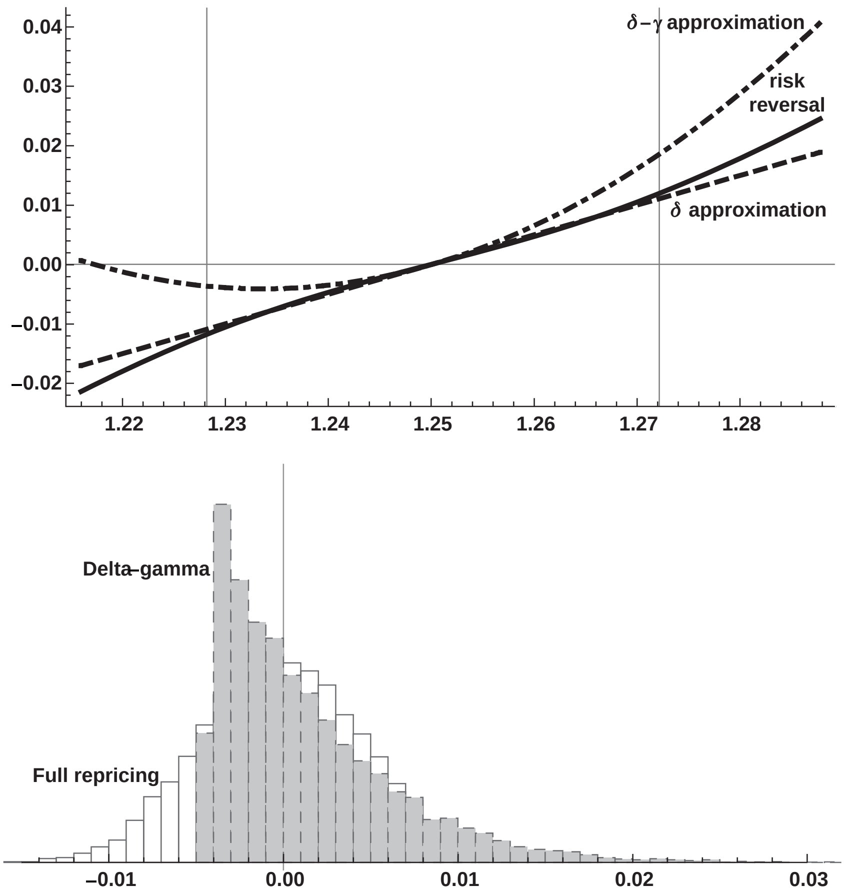

FIGURE 4.5 Delta-Gamma and Full-Repricing VaR for a Risk Reversal
Upper panel: One-day P&L in dollars of a risk reversal position as a function of the next-day USD-EUR exchange rate. The solid curve plots the exact P&L function, the dashed curve the delta approximation, and the short-long dashed curve the delta-gamma approximation.
Lower panel: Histogram of simulated P&Ls using the exact P&L function (unshaded bars), and the delta-gamma approximation (shaded bars).

The *delta* of any security, just as for an option, is the first derivative of the value $f(S_t, \tau)$ per unit or per share with respect to the risk factor S_t:

$$\delta_t \equiv \frac{\partial f(S_t, \tau)}{\partial S_t}$$

The *gamma* of the security is

$$\gamma_t \equiv \frac{\partial^2 f(S_t, \tau)}{\partial {S_t}^2}$$

As with options, in order to apply the delta-gamma approach without being misleading, the function $f(S_t, \tau)$ must be monotonic in S_t, that is, its first derivative does not change sign, no matter how high or low the value of the risk factor S_t. We assume that the risk factor S_t is lognormally distributed with zero mean. Recall that the parametric VaR for a long position in a single security is estimated as

$$\text{VaR}_t(\alpha, \tau)(x) = -\left(e^{z_* \sigma\sqrt{\tau}} - 1\right) x S_t$$

We now have to be careful with determining z_*, the standard normal ordinate corresponding to the VaR shock. If the delta is negative, then z_* is in the right rather than the left tail of the normal distribution. The most important example is VaR of a bond when we map bond prices to interest rates. We study this approach to bond VaR estimation in the rest of this chapter.

Under the delta-gamma approach, VaR is estimated as

$$\text{VaR}_t(\alpha, \tau)(x) = -x\left\{\left(e^{z_* \sigma\sqrt{\tau}} - 1\right) S_t \delta_t + \frac{1}{2}\left[\left(e^{z_* \sigma\sqrt{\tau}} - 1\right) S_t^2\right]^2 \gamma_t\right\}$$

where z_* is the ordinate of the standard normal distribution for

$$\left\{\begin{matrix} \alpha \\ 1-\alpha \end{matrix}\right\} \quad \text{for} \quad \delta_t \left\{\begin{matrix} < 0 \\ > 0 \end{matrix}\right\}$$

and α is the VaR confidence level. If $\delta_t > 0$, as in the examples up until now, then z_* is a negative number such as -2.33, and $e^{z_* \sigma\sqrt{\tau}} - 1 < 0$. If $\delta_t < 0$, as in the example we develop in the next section, then z_* is a positive number such as $+2.33$, and $e^{z_* \sigma\sqrt{\tau}} - 1 > 0$. Either way, the VaR estimate is a positive number.

4.2 YIELD CURVE RISK

This chapter on nonlinear risks may seem an odd place to discuss yield curve risk. The reason we do so is that, while bonds have a price expressed in currency units, they are more frequently viewed as exposures to interest rate rather than bond price fluctuations. Interest rate exposure is nonlinear, though much less drastically so than for options and other derivatives.

We start by focusing on default-free bonds, that is, bonds with deterministic future cash flows; their size and timing are known with certainty. The only thing that is risky about them is the discounted value the market will place on a dollar of cash flow delivered in the future. In subsequent chapters, we introduce credit spreads and discuss the risks of defaultable bonds when both risk-free and credit-risky interest rates can fluctuate. In practice, of course, as we noted in Chapter 2, no security is perfectly free of default risk.

Bond and other fixed-income security values are generally expressed for trading purposes as either a dollar/currency unit price, as a yield, or as a spread. However, none of these price metrics completely describe the *yield curve* the bond value depends on. Bond prices are not appropriate risk factors because

- Bonds vary in maturity. *Ceteris paribus*, longer-maturity bonds have a greater volatility than shorter-maturity bonds. In fact, as bonds near their maturity date, the volatility declines to zero, a phenomenon known as *pull-to-par*. So the risk of a bond changes merely because of the passing of time.
- Interest rates vary by the term over which the use of money is granted. Bond prices are determined, not by one interest rate, but by the yield curve, that is, interest rates of many different maturities. Interest rates are more truly the underlying risk factor than bond prices themselves. By focusing on the yield curve, we capture the differences in the sensitivity of a bond's value to changes in interest rates with different terms to maturity.

There are three common approaches to computing the VaR of a bond:

1. *Cash-flow mapping*. The value of any default-free bond can be decomposed into a set of zero-coupon bonds corresponding to each of the cash-flows associated with it. The prices of zero-coupon bonds can be used as risk factors. This approach can involve a large number of zero-coupon bonds if there are many cash flows and/or the bond has a long term to maturity. In such cases, we can bucket the cash flows into a smaller number of zero-coupon *nodes* or *vertexes*.
2. *Duration* of a bond can be measured and used together with the yield history of bonds of that maturity to derive an estimate of the bond's return volatility.

3. *Factor models* for bonds are focused on parsimonious sets of drivers of interest-rate risk. These can be interpreted statistically, for example, as *principal components* of interest rates, or economically, related to changes in the level, slope, and curvature of the yield curve.

The rest of this chapter introduces basic yield-curve concepts and focuses on the duration approach to measuring the interest-rate risk of a bond. Bonds denominated in a currency other than that of the portfolio owner's domicile have currency risk in addition to that generated by fluctuating interest rates.

4.2.1 The Term Structure of Interest Rates

This section gives a brief overview of yield curve concepts. The term structure can be described in three ways which have equivalent information content, the *spot* and *forward curves* and the *discount function*.

Any interest rate, including a spot rate, can be expressed numerically using any compounding interval, that is, the period during which interest accrues before interest-on-interest is applied. Interest rates with any compounding interval can be easily transformed into rates with any other compounding interval. Therefore, the choice of compounding interval is purely a choice of units, and is made for convenience only. We'll focus on continuously compounded rates, that is, interest rates expressed as though they were paid as a continuous flow, equal in every instant to the same fraction of the principal. The mathematics of the yield curve are simplest when expressed in continuously compounded terms, but real-world interest rates are rarely set this way. Interest rates are expressed as a rate per unit of time. We will express all interest rates at annual rates unless otherwise indicated.

We start with the notion of a *zero-coupon* or *discount bond*, a bond with only one payment, at maturity. We denote the time-t price per dollar of *principal amount* (also called *par* or *notional amount* or *face value*) of a discount bond maturing at time T by p_τ, where $\tau \equiv T - t$ is the time to maturity (a time interval rather than a date). The price at maturity of a risk-free discount bond is $p_0 \equiv 1$.

Spot and Forward Rates The *spot rate* r_τ is the continuously compounded annual rate of interest paid by a discount bond, that is, the rate of interest paid for the commitment of funds from the current date t until the maturity date T. The *continuously compounded spot rate* is the interest rate paid from time t to time T. The *continuously compounded spot* or *zero-coupon curve* is the function r_τ relating continuously compounded spot rates to the time to maturity or maturity date. This function changes as interest rates fluctuate. But for the rest of this section, we'll omit the "as-of" date subscript and keep only the time-to-maturity subscript, since our focus for now is on the properties of the yield curve at a point in time, rather than on its evolution over time.

The relationship between p_τ and r_τ is given by

$$\begin{aligned} p_\tau e^{r_\tau \tau} &= 1 \\ \Rightarrow \ln(p_\tau) + r_\tau \tau &= 0 \\ \Rightarrow \qquad\qquad r_\tau &= -\frac{\ln(p_\tau)}{\tau} \end{aligned} \tag{4.2}$$

that is, r_τ is the constant annual exponential rate at which the bond's value must grow to reach \$1 at time T.

Example 4.1 Let the price of a discount bond expiring in nine months be $p_{\frac{3}{4}} = 0.96$. The continuously compounded nine-month spot rate is

$$r_{\frac{3}{4}} = -\frac{\ln\left(p_{\frac{3}{4}}\right)}{\frac{3}{4}} = 0.054429$$

or 5.44 percent. In this, as in our other examples, we ignore the refinements of *day-count conventions*, which dictate how to treat such matters as variations in the number of days in a month or year, and the fact that the number of days in a year is not an integer multiple of 12.

A *forward rate* is the rate of interest paid for the commitment of funds from one future date T_1, called the *settlement date*, until a second future date T_2, called the *maturity date*, with $t \leq T_1 < T_2$. We now have two time intervals at work: the time to settlement $\tau_1 = T_1 - t$ and the time to maturity $\tau_2 = T_2 - T_1$. (Note that $T_2 = t + \tau_1 + \tau_2$.)

The *continuously compounded forward rate from time T_1 to time T_2*, denoted $f_{\tau_1,\tau_1+\tau_2}$, is the continuously compounded annual interest rate contracted at time t to be paid from time T_1 to time T_2. The *continuously compounded τ_2-period forward curve* is the function $f_{\tau_1,\tau_1+\tau_2}$ relating forward rates of a given time to maturity to the time to settlement or the settlement date. An example of a forward curve is the curve of three-month U.S. dollar money market rates implied by the prices of Chicago Mercantile Exchange (CME) Eurodollar futures contracts, for which we have $\tau_2 = \frac{1}{4}$.

We can relate continuously compounded forward rates to discount bond prices and spot rates. The forward rate is defined by

$$
\begin{aligned}
&\tfrac{1}{p_{\tau_1}} p_{\tau_1+\tau_2} e^{f_{\tau_1,\tau_1+\tau_2}\tau_2} = 1 \\
\Rightarrow\quad &\ln\left(\tfrac{p_{\tau_1+\tau_2}}{p_{\tau_1}}\right) + f_{\tau_1,\tau_1+\tau_2}\tau_2 = 0 \\
\Rightarrow\quad &f_{\tau_1,\tau_1+\tau_2} = -\tfrac{1}{\tau_2}\ln\left[\tfrac{p_{\tau_1+\tau_2}}{p_{\tau_1}}\right] = \tfrac{1}{\tau_2}\ln\left[\tfrac{p_{\tau_1}}{p_{\tau_1+\tau_2}}\right]
\end{aligned}
\tag{4.3}
$$

The first line of Equation (4.3) defines the forward rate as the constant rate at which the price of a bond maturing on the forward maturity date must grow so as to equal the price of a bond maturing on the forward settlement date. From the last line of Equation (4.3), together with the last line of (4.2), we see that forward rates can also be calculated directly from spot rates:

$$f_{\tau_1,\tau_1+\tau_2} = \frac{r_{\tau_1+\tau_2}(\tau_1+\tau_2) - r_{\tau_1}\tau_1}{\tau_2}$$

The *instantaneous forward rate* with settlement date T, denoted f_τ, is the limit, as $T_1 \to T_2$, of $f_{\tau_1,\tau_1+\tau_2}$:

$$f_\tau = \lim_{\tau_2\to 0} f_{\tau_1,\tau_1+\tau_2} = \lim_{\tau_2\to 0}\frac{r_{\tau_2}(\tau_1+\tau_2) - r_{\tau_1}\tau_1}{\tau_2} = r_\tau + \frac{dr_\tau}{d\tau}\tau$$

after simplifying notation by setting $\tau \equiv \tau_1$. The instantaneous forward rate is the interest rate contracted at time t on an infinitely short forward loan settling τ periods hence. For concreteness, one can think of it as a forward on the overnight rate prevailing at time T. The *instantaneous forward curve* is the function f_τ relating instantaneous forward rates to the time to settlement or the settlement date.

A forward rate with a finite time to maturity can be viewed as the average of the instantaneous forward rates over the time to maturity. Integrating over a range of settlement dates, we have

$$f_{\tau_1,\tau_1+\tau_2} = \frac{1}{\tau_2}\int_{\tau_1}^{\tau_1+\tau_2} f_s ds$$

A τ-year continuously compounded spot rate, that is, the constant annual rate at which a pure discount bond's value must grow to reach one currency unit at time T, can be expressed by integrating the instantaneous forward curve over the time to maturity:

$$r_T = \frac{1}{T}\int_0^T f_t dt$$

The relationship between spot and forward rates has an important implication: the forward curve is higher than the spot curve for maturity intervals in which the spot curve is upward sloping, and vice versa.

Spot curves and forward curves are two equivalent ways of expressing the term structure of interest rates as a function of the time to maturity. A third is the discount function or *discount factor curve*, which relates the prices of zero-coupon bonds to their times to maturity. Unlike either spot curves or forward curves, the discount function has no compounding intervals. As we saw in defining them, forward rates can be viewed as logarithmic changes along the discount function. Discount factors are very close to 1 for short maturities, and close to 0 for long maturities. The discount function must slope downwards.

4.2.2 Estimating Yield Curves

None of the yield curve concepts we've just defined are, in general, directly observable. This is a pity, because modeling and implementing risk measures for fixed-income securities requires them. Apart from a small number of short-term bonds such as U.S. Treasury bills and strips, there are not many zero-coupon bonds. And while many bond prices are expressed in yield terms, money-market instruments are among the only single cash-flow securities that would permit one to easily convert the yield into a continuously compounded spot rate. Forward rates are expressed in money market futures and *forward rate agreements*, the OTC analogue of money market futures.

In other words, yield curves have to be extracted or estimated from the actually traded mishmash of diverse fixed-income security types. Aside from the diversity of cash flow structures and quoting conventions, there are other problems with the market data used in estimating yield curves, for example,

Liquidity. Different securities on what seems to be the same yield curve can have very different liquidity, so that effectively, their prices are generated by different yield curves. For example, *on-the-run* or freshly issued U.S. Treasury notes have lower yields and higher prices than *off-the-run* notes, that is, issues from less recent auctions, which tend to have lower prices and a liquidity premium.

Embedded options. Some bonds are callable or have other option-like features. In fact, some U.S. Treasury notes issued prior to 1985 were callable. These options can have value, and if they do, their prices do not coincide with the discount factors for similar bonds without options.

Taxes. Different types of bonds have different tax treatment. For example, in the United States, income from U.S. Treasury issues is not taxed at the state and local level, and the income from most bonds issued by state and local governments is not taxed at all, while income from corporate bonds is taxable at the federal, state, and local level. These tax differences have a large impact on prices that has to be taken into account in estimating yield curves.

To get around these problems, one can filter the data so that they are uniform with respect to their tax, liquidity, and optionality characteristics. Alternatively, one can estimate the impact of these characteristics and adjust the data for them.

Another set of issues making it difficult it construct yield curves is that the estimated yield curves can behave strangely in several ways:

Asymptotic behavior. If we extrapolate yield curves to much longer maturities than the data provide, in itself a problematic exercise, the prices or spot rates may become negative or infinite.

Violations of no-arbitrage conditions. Estimated yield curves may display discontinuities, spikes, and other oddities. Intermediate points on the spot and forward curves falling between actual observations may then be much higher or lower than neighboring points. Lack of smoothness may lead to the apparent possibility of instantaneous arbitrage between bonds or forward contracts with different maturities as computed from the curve.

The discount, spot, and forward curves are different ways of expressing the same time value of money, so any of these forms of the yield curve can be estimated and transformed into any of the others, with the form chosen for a particular purpose a matter of convenience or convention.

Bootstrapping and Splines A common approach "connects the dots" between observed yields, spot rates, or forward rates:

Bootstrapping. Each security is stripped down to its individual cash flows, which are arranged in maturity order. Starting with the shortest maturity, and using the results of each step to support the subsequent step, the discount factors or spot rates corresponding to each maturity are computed. Futures and forwards as well as cash securities can be included.

Spline interpolation is a process for connecting data points by passing polynomial functions through them. Polynomials have the advantage that, depending on their degree, they lead to smooth curves in the sense that they have finite derivatives. For example, cubic splines have finite first, second, and third derivatives.

The bootstrapping and spline approaches have the advantage that the price or yield of each of the securities used to estimate the yield curve can be recovered exactly from the estimated curve. But both approaches tend to produce curves that are spiky or excessively wavy, or extrapolate out to infinity or zero. In practice, these problems can be addressed by preprocessing the data or by additional smoothing techniques.

Parametric Estimates Parametric approaches begin with a model that limits the forms the yield curve can take. For example, the *Nelson-Siegel* specification of the instantaneous forward rate is similar in spirit to a delta-gamma approach and is given by

$$f(\tau;\beta_0,\beta_1,\beta_2,\theta) = \beta_0 + \beta_1 e^{-\frac{\tau}{\theta}} + \beta_2 \frac{\tau}{\theta} e^{-\frac{\tau}{\theta}} \tag{4.4}$$

where $(\beta_0,\beta_1,\beta_2,\theta)$ is a vector of parameters to be estimated and τ, representing the time to maturity, is the single argument of the function. The corresponding representation of the spot rate is the definite integral of this instantaneous forward rate over τ, Equation (4.4) or

$$\begin{aligned} r(\tau;\beta_0,\beta_1,\beta_2,\theta) &= \beta_0 + (\beta_1+\beta_2)\left(\tfrac{\tau}{\theta}\right)^{-1}\left(1-e^{-\frac{\tau}{\theta}}\right) - \beta_2 e^{-\frac{\tau}{\theta}} \\ &= \beta_0 + \beta_1\left(\tfrac{\tau}{\theta}\right)^{-1}\left(1-e^{-\frac{\tau}{\theta}}\right) \\ &\quad + \beta_2\left[\left(\tfrac{\tau}{\theta}\right)^{-1}\left(1-e^{-\frac{\tau}{\theta}}\right) - e^{-\frac{\tau}{\theta}}\right] \end{aligned}$$

With different values of the parameters, the function is capable of fitting a wide range of typical yield curve shapes. Each term and parameter in the function contributes a distinct element to these typical patterns:

- β_0 is the asymptotic value of the forward rate function and represents the forward rate or futures price prevailing at very long maturities.
- As $\tau \downarrow 0$, the forward and spot rates tend toward $\beta_0+\beta_1$, which thus more or less represents the overnight rate. To constrain the very

short-term interest rate to nonnegative values, the condition $\beta_1 > -\beta_0$ is imposed. If $\beta_1 > 0$ (< 0), the very short-term value of the function is higher (lower) than the long-term value.

- The term $\beta_1 e^{-\frac{\tau}{\theta}}$ imposes exponential convergence to the long-term value β_0. If $\beta_1 > 0$ (< 0), the convergence is from above (below).
- The term $\beta_2 \frac{\tau}{\theta} e^{-\frac{\tau}{\theta}}$ permits hump-shaped behavior of the yield curve. If $\beta_2 > 0$ (< 0), the function rises above (falls below) its long-term value before converging.
- The speed of convergence of both the simple and humped exponential terms to the long-term value is governed by θ. A higher value of θ corresponds to slower convergence. That is, for a given gap between very short- and long-term rates, a higher value of θ lowers any other term spread.

We use the Nelson-Siegel specification in the examples in the rest of this chapter. The reason is that the parameters have a nice interpretation as three yield curve risk factors:

1. β_0 is the yield curve *level* factor and has a constant factor loading of unity.
2. β_1 is the yield curve *curvature* factor. It has a factor loading of $\left(\frac{\tau}{\theta}\right)^{-1}\left(1 - e^{-\frac{\tau}{\theta}}\right)$.
3. β_2 is a factor representing an "overshoot-and-converge" pattern in the yield curve level, and has a factor loading of $\left(\frac{\tau}{\theta}\right)^{-1}\left(1 - e^{-\frac{\tau}{\theta}}\right) - e^{-\frac{\tau}{\theta}}$.

In particular, by changing β_0, we can induce parallel shifts up or down in the yield curve in a simple way. We use this property in the rest of this chapter. The Nelson-Siegel is used more in academic work than in the practical work of financial intermediaries, since it doesn't exactly replicate the input security prices. But it is a very practical tool for explicating risk measurement techniques for fixed income.

Figure 4.6 illustrates the Nelson-Siegel function. It is estimated using unweighted least squares and data on Libor, eurodollar futures, and plain-vanilla swap rates on Nov. 3, 2007.

4.2.3 Coupon Bonds

A coupon bond is an interest-bearing instrument that makes regular payments, called *coupons*, at contractually specified times $t_1 < t_2 < \ldots < t_n = T$, and a final payment of the principal or face value at maturity date T. For most coupon bonds, the coupons are all the same amount c per dollar of

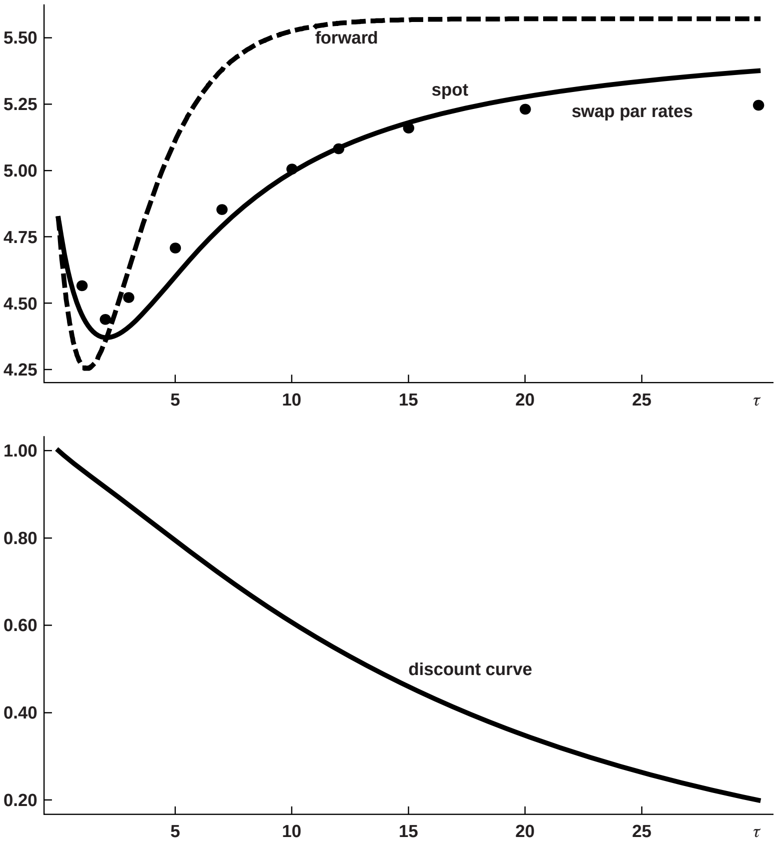

FIGURE 4.6 Spot, Forward, and Discount Curves
Three equivalent representations of the yield curve, as spot, forward, and discount curves, USD swaps, Nov. 1, 2007, maturities in years.
Upper panel: Continuously compounded spot curve and corresponding instantaneous forward curve, interest rates in percent. The dots represent the U.S. dollar swap par rate prevailing on Nov. 3, 2007.
Lower panel: Discount curve, expressed as dollars per dollar of face value or as a decimal fraction of face value.

face value, and the payment intervals are all equal: $t_{i+1} - t_i = t_i - t_{i-1} = h$, $i = 1, 2, \ldots, n-1$. We'll denote by $p_{\tau,h}(c)$ the price at time t of a bond maturing at time T, with an annual coupon rate c, paid $\frac{1}{h}$ times annually.

For a newly issued bond, the first coupon payment is h years in the future, so $t_1 = t + h$ and $T = nh$. For example, a 10-year bond with semiannual coupons has $n = \frac{10}{\frac{1}{2}} = 10 \cdot 2 = 20$ coupon payments. Not only the time to maturity of the bond, but also the coupon and the frequency of coupon payments affect the bond's value.

A coupon bond can be viewed as a package of discount bonds, with each coupon payment and the principal payment at maturity is viewed as a discount bond. This lets us relate coupon bond prices to the continuously compounded spot curve:

- The i-th coupon payment is equivalent to an $i \times h$-year discount bond with a face value of c. If it were traded separately, its price would be $c \times e^{-r_{ih} ih}$.
- The principal payment is equivalent to a T-year discount bond with a face value of \$1. If it were traded separately, its price would be $e^{-r_\tau \tau}$.

The value of the coupon bond is the arithmetic sum of these values. This identity can be expressed in terms of spot rates:

$$p_{\tau,h}(c) = ch \sum_{i=1}^{\frac{\tau}{h}} e^{-r_{ih} ih} + e^{-r_\tau \tau}$$

The continuously compounded *yield to maturity* of a coupon bond $y_\tau(c)$ is defined by

$$p_{\tau,h}(c) = ch \sum_{i=1}^{\frac{\tau}{h}} e^{-y_\tau(c) ih} + e^{-y_\tau(c)\tau} \tag{4.5}$$

The yield can be interpreted as the constant (over maturities) spot rate that is consistent with the value of the bond. There is a one-to-one relationship between the price and the yield of a coupon bond, given a specific coupon, maturity, and payment frequency. Given a price, the formula can be solved for $y_\tau(c)$ using numerical techniques. We can therefore express the bond price as $p(y_t)$, expressed as a percent of par (or dollars per \$100 of notional value), with specific parameters τ, h, and c.

A *par bond* is a bond trading at its face value \$1. For a par bond, the yield to maturity is equal to the coupon rate.

Example 4.2 Let's price a 10-year coupon bond using the yield curve represented in Figure 4.6. We assume the bond has annual payments ($h = 1$) of 5 percent ($c = 0.05$). The spot rates and discount factors are:

Maturity	Spot rate r_t	Discount factor	Coupon PV
1	4.4574	0.95640	0.04782
2	4.3702	0.91631	0.04582
3	4.4083	0.87612	0.04381
4	4.4967	0.83538	0.04177
5	4.5989	0.79458	0.03973
6	4.6983	0.75435	0.03772
7	4.7881	0.71522	0.03576
8	4.8666	0.67751	0.03388
9	4.9342	0.64142	0.03207
10	4.9919	0.60702	0.03035

The last column displays the present value of the coupon payment for each maturity, expressed in dollars per \$1 of face value. The sum of these present values is 0.388716. Including the present value of the principal at maturity, we have

$$\begin{aligned} p_{\tau,h}(c) &= 0.05 \sum_{t=1}^{10} e^{-r_t t} + e^{-r_\tau \tau} \\ &= 0.388716 + 0.60702 \\ &= 0.995737 \end{aligned}$$

per \$1 of face value, or 99.5737 par value. The bond's yield to maturity is 4.9317 per annum.

A simple example of a coupon bond curve is the swap curve. For U.S. dollar–denominated swaps, the compounding convention is semiannual, while for most other currencies it is annual. For a flat swap curve, that is, a swap curve on which swap rates for any maturity are equal to a constant, there is a simple formula for the spot rate:

$$r_t = h \log(r^s) \qquad \forall t > 0$$

where r^s is the swap rate. For example, with a flat semiannual swap curve of 3.5 percent, the spot rate is a constant 3.470 percent.

4.3 VAR FOR DEFAULT-FREE FIXED INCOME SECURITIES USING THE DURATION AND CONVEXITY MAPPING

We have now seen that a fixed-income security can be decomposed into a set of cash flows occurring on specific dates in the future and that its value can be computed as the value of that set of discount bonds. We can therefore measure the VaR and other distributional properties of a fixed-income security using the distributions of its constituent discount bonds; the return distribution of the bond is the return distribution of the portfolio of zeroes. The security is treated as a portfolio consisting of the discount bonds. To carry out this cash-flow mapping approach to computing VaR, we require time series of returns on all the discount bonds involved. From these time series, we can compute the volatilities and correlations needed for the parametric and Monte Carlo approaches, and the historical security returns needed to carry out the historical simulation approach.

This section lays out a simpler approach to measuring VaR for a bond using the *duration-convexity* approximation. It is a specific application of the delta-gamma approach to VaR measurement. The single risk factor in this case is the yield to maturity of the bond. It is straightforward to apply delta-gamma to coupon bonds, since bond values are monotonically decreasing functions of yield.

We will treat the price of the bond as a function of its time-t yield to maturity y_t. In the notation of this chapter, $p(y_t)$ plays the role of the general function $f(S_t, \tau)$, but we will ignore explicit dependence on the time to maturity other than through the yield. For fixed-income securities with very short times to maturities, this would introduce a material bias, but "roll-down" can be ignored for most bonds. We assume that $p(y_t)$ can be differentiated twice with respect to y_t.

By reducing the number of factors that influence bond prices to a single yield rather than an entire term structure of interest rates, this approach implicitly assumes that any change in bond value is caused by a *parallel shift* in the yield curve. This approach thus ignores the impact of changes in the shape of the yield curve to which the yield to maturity is invariant. A curve steepening or flattening that leaves the level of yield unchanged may impact the value of the bond, so this duration-convexity approximation can understate risk.

In order for us to use the yield as a risk factor, we need an additional ideal condition, namely, the existence of a liquid market in which freshly issued bonds of precisely the same maturity trade daily. This condition is not always met for the plain-vanilla swap market as well as the government bond market, as we see in Chapters 12 and 14.

With all these caveats, the duration-convexity approach is a reasonably accurate approximation for most bonds. It is also relatively easy to compute, and it is intuitive for fixed-income traders. For these reasons, it has become quite standard.

4.3.1 Duration

We start by defining two related concepts, the DV01 and modified duration of a bond. The *DV01* is the change in value that results from a one basis point (0.0001) change in yield. It is multiplied by −1 for the convenience of working with a positive number:

$$\mathrm{DV01} \equiv -dy_t \frac{dp}{dy_t} = -0.0001 \frac{dp}{dy_t}$$

The concept of DV01 applies to any interest-rate sensitive security, including options. It can be defined to encompass changes in any yield curve concept, such as spot or forward rates, as long as the change is a parallel shift, that is, a uniform 1bp shift of the entire curve. We can get the DV01 of the position by multiplying the quantity we just defined by the par amount of the security.

In many textbooks on fixed income, DV01 and duration are calculated by algebraically differentiating Equation 4.5, which defines the relationship between bond price and yield with respect to the yield. Nowadays, it is generally easier to use numerical approaches. The DV01 of a bond can be easily and accurately calculated as the difference in the value of the coupon bond with the entire yield curve shifted up and down, in parallel, by 0.5bp:

$$\begin{aligned} \mathrm{DV01} \approx \Delta p &= -0.0001 \frac{p(y_t + 0.00005) - p(y_t - 0.00005)}{0.0001} \\ &= p(y_t - 0.00005) - p(y_t + 0.00005) \end{aligned}$$

where Δp has been specified as the change for a 1bp change in yield.

The *modified duration* of a bond is defined as

$$\mathrm{mdur}_t \equiv -\frac{1}{p}\frac{dp}{dy_t} = \frac{1}{p}\frac{1}{dy_t}\mathrm{DV01} \tag{4.6}$$

DV01 is expressed in dollars per \$100, that is, dollars per par value of the bond, per 1bp of yield change, while modified duration is a proportional measure, specifically, the percent change in the bond's value for a 1 percent

(100 basis point) change in yield. Like DV01, modified duration is usually computed numerically:

$$\text{mdur}_t \approx -\frac{1}{p}\frac{\Delta p}{\Delta y_t} = \frac{1}{p}\frac{\text{DV01}}{0.0001}$$

Example 4.3 (DV01 and Duration) We illustrate the duration and convexity approach by continuing Example 4.2 of a default-free plain vanilla 10-year "bullet" bond paying an annual coupon 5 percent and priced using the yield curve displayed in Figure 4.6. To make notation a bit easier and avoid lots of zeroes, we'll express bond par values as $100.

The DV01 is $0.080466 per $100 of notional value:

$$\begin{aligned}\text{DV01} &= -0.0001\frac{99.5335175 - 99.6139834}{0.0001} = 99.6139834 - 99.5335175 \\ &= 0.080466\end{aligned}$$

The modified duration of the bond in our example is 8.08104. So if the interest rate falls by 1 percentage point, the value of the bond will rise by approximately 8.08 percent. If the interest rate falls by 1 basis point, the value of the bond will rise by approximately 0.0808 percent. A $1,000,000 notional value position in the bond will decline by $804.66 per basis point: At a price of 99.5737, the position value is $995,737, relative to which the decline in value is $804.66.

4.3.2 Interest-Rate Volatility and Bond Price Volatility

To compute the VaR, we assume we have a trusted estimate $\hat{\sigma}_t$ of the volatility of daily changes in the bond yield. This is called a yield or *basis point volatility*, as opposed to return or price volatility.

There are two generally accepted ways to compute interest-rate volatility:

Yield volatility. In this definition, we treat the yield as though it were a price and state the volatility as the standard deviation of proportional changes in the yield. For example, if the yield level is 5 percent and the yield volatility is 15 percent, then the annualized standard deviation of yield changes is $0.15 \times 0.05 = 0.0075$ or 75 basis points.

Yield volatility, like return volatility, can be historical, based on historically observed yields, or implied, based on fixed-income option prices. When discussing implied rather than historical volatility, yield volatility is often called *Black volatility*. Prices of OTC interest rate options such as swaptions, are typically quoted as Black vols, although there are readily available screens on Reuters and other market information providers that translate the entire grid of option and underlying swap maturities and maturities into basis point volatilities as well as dollar prices per option unit.

In expressing these interest-rate option prices, the yield is treated as lognormally distributed. As in the case of other option implied volatilities, this is not so much an authentic modeling assumption as a pricing convention that can be translated into currency unit prices via the Black-Scholes formulas. The empirical underpinning for the lognormality assumption is even weaker than for asset prices. However, it does, at least, prevent negative interest rates from appearing.

Basis-point volatility. In this definition of volatility, we state the volatility of changes in the yield itself, equal to $y_t\sigma_y$. In the example we just gave, the basis-point volatility corresponding to a yield level of 5 percent and an annual yield volatility of 15 percent is 75 basis points per annum. It is, however, generally not expressed in annual terms, but rather at a daily rate, using the square-root-of-time rule. With a day count of 252 days per year, we have a daily basis-point volatility of 4.72 basis points.

The same choice of definitions applies to the computation of any across-the-curve interest-rate volatility.

The relationship between bond price volatility σ_p and yield volatility σ_y is derived from the definition of modified duration, which we can rewrite as

$$\frac{dp}{p(y_t)} = -\,\text{mdur}_t\, dy_t$$

As can be seen in Figure 4.7, the relationship is nearly linear. When volatility is defined as yield volatility, the change in yield, measured in interest-rate units such as basis points, is $dy_t = \sigma_y y_t$, so

$$\sigma_p = \text{mdur}_t\, y_t\sigma_y$$

This expression for bond price volatility will be used in the VaR computation examples that follow.

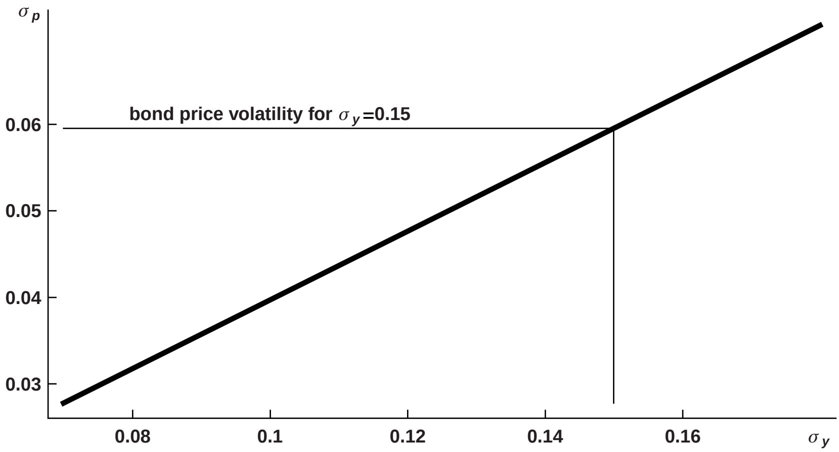

FIGURE 4.7 Bond Price and Yield Volatility
Bond price volatility is a linear function of yield volatility for a given term structure. Volatilities are expressed as decimals at an annual rate.

Example 4.4 (Yield and Bond Price Volatility) Continuing Example 4.3 of a default-free plain vanilla 10-year "bullet" bond paying an annual coupon of 5 percent, let the Black (yield) volatility equal 15 percent. The price volatility is then

$$\sigma_p = \text{mdur}_t\, y_t \sigma_y = 8.08104 \times 0.04932 \times 0.15 = 0.0598$$

or 5.98 percent.

4.3.3 Duration-Only VaR

We'll begin by calculating VaR for the bond position using duration only. This is essentially a delta approximation. For many purposes, using duration only is accurate enough.

Given an estimate of the yield volatility, whether based on historical data or an implied volatility, we can say that, with confidence level α, the change in yield over the period t to $t + \tau$ will be less than

$$\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t$$

with a probability of α, where z_* is the ordinate of the standard normal distribution at which $\Phi(z) = \alpha$.

A VaR estimate for a long position of x units of the bond is then

$$\text{VaR}_t\,(\alpha, \tau) = x\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t \times \text{mdur}_t \times p(y_t)$$

where x is the par value of the bonds. The VaR is thus equal to the size of the position times the absolute value of the decline in price in the VaR scenario. This can be equivalently expressed as

$$\text{VaR}_t\,(\alpha, \tau) = x\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t \times \text{DV01}$$

This is a simpler expression, but isn't typically encountered because modified duration is the more common metric for expressing bond price sensitivity.

Note that we take the ordinate of α rather than $1 - \alpha$. Why? In the terminology of the delta-gamma approach, we identify

$$\frac{dp}{dy_t} = \delta_t = -p\,\text{mdur}_t$$

as the delta of the bond. Since $\delta_t < 0$, we have used the right-tail rather than the left-tail ordinate of the standard normal distribution. This corresponds to the fact that the bond loses value when the yield rises.

Example 4.5 (Duration-Only VaR) In our standing example, the market parameters for the estimate are

Initial notional value	\$1,000,000
Initial market value	\$995,737
Initial yield	4.9317%
mdur	8.08104 bp per bp of yield
Yield vol σ_y	15% p.a. (0.945% per day)

The VaR parameters are

Time horizon	1 day
Confidence level	99%
z_*	2.33

The duration-only VaR is then:

$$\begin{aligned}\text{VaR}_t(\alpha, \tau) &= x\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t \, \text{mdur}_t \, p(y_t) \\ &= 10^6 \times (e^{0.02198} - 1) \times 0.04932 \times 8.081 \times 0.995737 \\ &= 10^6 \times 0.022253 \times 0.396835 \\ &= 8819.78\end{aligned}$$

4.3.4 Convexity

We can make the VaR estimate somewhat more precise by approximating the bond's nonlinear exposure to yield. To do this, we measure the bond's *convexity*, the second derivative of its value with respect to the yield, normalized by the price:

$$\text{conv}_t \equiv \frac{1}{p(y_t)} \frac{d^2 p}{dy_t^2}$$

Like DV01 and duration, convexity can be computed for all interest-rate sensitive securities and using across-the-curve interest-rate concepts other than yield to maturity.

Convexity is always positive for plain-vanilla bonds, but it can be negative for some structured products. Mortgage-backed securities are an important example of bonds with negative convexity. We will see some structured credit examples in Chapters 9 and 11, and an example of the difficulty of managing negative convexity in Chapter 14.

The convexity of a bond, like the DV01, can be computed numerically by shifting the yield curve up and down, in parallel, by 0.5 bp, twice. We have

$$\begin{aligned}\Delta^2 p \equiv \Delta[\Delta(p)] &= \Delta[p(y_t + 0.00005) - p(y_t - 0.00005)] \\ &= p(y_t + 0.0001) - p(y_t) - [p(y_t) - p(y_t - 0.0001)] \\ &= p(y_t + 0.0001) + p(y_t - 0.0001) - 2p(y_t)\end{aligned}$$

This computation is identical to measuring the bond's DV01 for yields that are 0.5 basis points higher and lower than the current yield, and taking their difference.

Convexity is then measured as

$$\text{conv}_t \approx \frac{1}{p} \frac{\Delta^2 p}{\Delta y_t^2} = \frac{1}{0.0001^2 p}[p(y_t + 0.0001) + p(y_t - 0.0001) - 2p(y_t)]$$

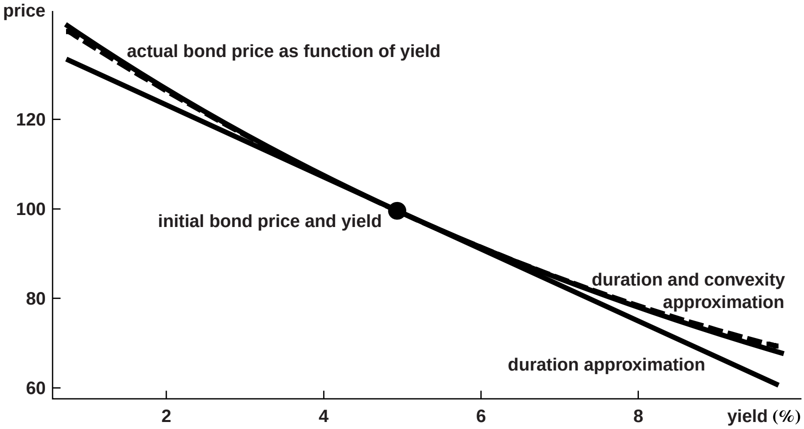

FIGURE 4.8 Approximating the Bond Price-Yield Relationship
Using duration alone provides a linear approximation to the sensitivity of the bond price to changes in yield. For small changes in yield, this is fairly close. Using convexity provides a linear-quadratic approximation. It's an improvement, but still not quite exact.

Example 4.6 (Convexity) Continuing Example 4.5, the duration and convexity approximation is illustrated in Figure 4.8. The convexity of the bond is 74.2164. We can compute this result as follows. The DV01 measure at a yield 0.5 basis points lower (higher) than the current yield is 0.0805029 (0.0804290). The difference between these DV01s is 0.0000739. Dividing this result by $0.0001^2 \times p$ gives the result for convexity.

There is an alternative convention for expressing duration. It is in fact more widely used than the one presented here, appearing for example on Bloomberg bond analysis screens and in the published research of most banks and brokerages; the textbooks are mixed. This alternative convention doesn't affect duration, but does make a difference for convexity. It expresses the yield as a decimal ($1\text{bp} \equiv 0.0001$) in the pricing formula (there is no alternative), but as a percent in the denominator, so $1\text{bp} \equiv 0.01$. The DV01 or Δp or $\Delta^2 p$ we just defined is multiplied by 100. In the alternative convention, modified duration is the same, but convexity is expressed in units one-hundredth the size of those here.

4.3.5 VaR Using Duration and Convexity

We can now apply the full delta-gamma approach to compute VaR for a bond. The gamma is represented by convexity:

$$\frac{d^2 p}{dy_t^2} = \gamma = p \operatorname{conv}_t$$

With duration and convexity, we have a linear-quadratic or second-order approximation to the bond's value:

$$\Delta p \approx -\text{mdur}_t\, \Delta y_t + \frac{1}{2}\text{conv}_t(\Delta y_t)^2$$

The convexity term increases the gain from a decline in yield and reduces the loss from a rise in yield. The VaR estimate for the bond is now

$$\text{VaR}_t\,(\alpha, \tau) = x\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t\, p\, \text{mdur}_t - \frac{1}{2} x \left[\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t\right]^2 p\, \text{conv}_t$$

Example 4.7 (VaR for a Default-Free Plain-Vanilla Coupon Bond) The additional parameter, compared to the previous example of duration-only VaR, is

conv_t	74.2164bp per squared bp of yield

The convexity adjustment, which attenuates the loss in the VaR scenario and is therefore subtracted from the linear loss term, is:

$$\begin{aligned}
& x\frac{1}{2}\left[\left(e^{z_* \sigma_y \sqrt{\tau}} - 1\right) y_t\right]^2 \text{conv}_t\; p(y_t) \\
&= 10^6 \times \frac{1}{2}(0.022253 \times 0.04932)^2 \times 74.2164 \times 0.995737 \\
&= 10^6 \times \frac{1}{2} \times 1.20141 \times 10^{-6} \times 74.2164 \times 0.995737 \\
&= 44.39
\end{aligned}$$

The VaR is thus

$$\text{VaR}_t\left(0.99, \frac{1}{252}\right) = 8,819.78 - 44.39 = 8,775.39$$

FURTHER READING

Taleb (1997) and Hull (2000) are textbook introductions to option modeling and risk management. Allen (2003) is a general risk management textbook with a strong focus on derivatives. See Rubinstein (1994) on derivatives and nonlinear risk.

VaR for nonlinear portfolios is discussed by Britten-Jones and Schaefer (1999). Alternative ways of carrying our delta-gamma are explored in Mina and Ulmer (1999), and methods for speeding up simulations when repricing are discussed in Mina (2000).

Tuckman (2002) is a textbook covering fixed-income modeling. Shiller and McCulloch (1990) provides a compact but accessible introduction to term-structure concepts. The yield-curve fitting technique employed in this chapter was originally developed by Nelson and Siegel (1987). The interpretation of the Nelson-Siegel approach as a factor model is presented in Diebold and Li (2006). See Chance and Jordan (1996) on the duration-convexity approach. Cash flow mapping alternatives to duration-convexity VaR are discussed in Mina (1999).